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Dynamical versus kinematical symmetry breaking revisited

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Abstract

We reinvestigate the question of whether or not the dynamical symmetries of the two-dimensional interacting boson model can be transformed into each other by the quantum deformation of the algebraic structure. The deformation of both the vibrational and the rotational spectra is considered, and the breaking of the classical symmetry is measured quantitatively, while a systematic search for the best fit is performed. In no case can the quantum-deformed spectrum reach the other classical limit; only a modest improvement is observed with the phase deformation of the rotational spectrum.

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1. Introduction

The appearance of quantum algebras initiated very lively research on their possible application in many branches of physics, including nuclear spectroscopy [1]. Quantum algebras are deformed versions of the classical Lie algebras; therefore, the symmetries based on them are more general than those appearing in Lie-algebraic models. One of the interesting questions one can address is: Can the quantum deformation of the algebraic structure (in other words, the kinematical breaking of the symmetry) bridge the gap between the dynamical symmetries of the algebraic models? These symmetries, described by a nested chain of Lie subalgebras, show up as analytically solvable limits of the models, and correspond to well-defined physical situations, e.g. the rotational or vibrational behaviour of a many-body system. The regions in between the dynamical symmetries, however, need numerical treatment in the usual approach. On the other hand, quantum algebraic descriptions provide us with analytical solutions even if we move away from the classical Lie-algebraic symmetry. The question is: How far can one go in this way along the path to the other classical limit?

The problem of the complete breaking of dynamical symmetries, i.e. the bridging between the classical limits by quantum deformation, has been investigated in one- [2], two- [3, 4] and three-dimensional [5, 6] models, and is reviewed in [1]. The conclusions of these works

are not univocal, which is related to the fact that the applied methods are different. In some cases only the energies of the ground-state rotational band are investigated, while in others the complete spectrum, including electromagnetic transitions, is treated. Furthermore, the allowed regions of the deformation parameters were also different. Due to this reason, the available results are still not conclusive.

In the present paper we reinvestigate the question of kinematical versus dynamical symmetry breaking within the framework of the two-dimensional interacting boson model of an $SU(3)$ group structure [7]. This model is a toy version of the three-dimensional interacting boson model (IBM) [8], which has a $U(6)$ group structure, and turned out to be very successful in the description of the collective motion of real nuclei. The two-dimensional model shows the basic features of the real IBM in a simplified form; therefore, it seems to be very well suited for our study. It is simple enough in order to study its quantum deformation, yet it is not trivial: it contains two dynamical symmetries, corresponding to the rotational and vibrational limits. We consider the complete spectrum here, including electromagnetic transitions, similar to the work [3], and not only the energies of the states in the ground-state band, like in [4]. The novel features of the present work in comparison with the previous ones are as follows. (i) Here we measure quantitatively the similarity of the quantum-deformed spectrum to those of the corresponding classical Lie-algebraic dynamical symmetries. The conclusions so far have been based on qualitative comparisons between the spectra. (ii) We perform a systematic search in fitting the kinematically broken spectrum to those with exact dynamical symmetries. (iii) We investigate the problem from both sides, i.e. trying to reach the classical rotational spectrum with the quantum-deformed vibrational one, and vice versa, based on more recent works [1]. (At the time of the earlier works, only the deformation of the vibrational $SU(2)$ was known [9].) As for the deformation parameter, one has to distinguish between two cases. (i) When it is a real number, or a pure phase value, then the quantum-deformed model is well established in the sense that it has real physical quantities. (ii) When it is an arbitrary complex number, this is not the case; therefore, the physical interpretation of a model with such an algebraic structure is not yet clear, in spite of some pioneering work along this line [1]. Nevertheless, we also investigate this possibility in order to see what kind of improvement can be made (formally), if any.

The structure of this paper is as follows. In section 2 we review very briefly first the classical Lie-algebraic model, based on [7]; then we consider the relevant formulae of its quantum deformation, based on [1]. In section 3 we present our results and finally the conclusions are drawn in section 4.

2. The model

2.1. Classical algebraic description

In the two-dimensional interacting boson model the building blocks are bosons with angular momentum $m = 0, \pm 2$. Their creation $\left(a_0^\dagger, a_+^\dagger, a_-^\dagger\right)$ and annihilation (a_0, a_+, a_-) operators satisfy the usual boson commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \qquad [a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{1}$$

The physical operators are obtained in terms of their particle number conserving bilinear products:

$$\Lambda_{ij} = a_i^\dagger a_j, \tag{2}$$

which satisfy the

$$[\Lambda_{ij}, \Lambda_{kl}] = \delta_{jk}\Lambda_{il} - \delta_{il}\Lambda_{jk} \tag{3}$$

commutation relations, and generate the $U(3)$ group. (Here we follow the notations of most of the papers related to the interacting boson model, and refer to the groups rather than to their algebras.) The

$$N = \sum_i \Lambda_{ii} \tag{4}$$

particle number operator is kept constant; therefore, the number of independent operators is 8, and the underlying group structure of the model is $SU(3)$. Only totally symmetric irreducible representations (irreps) $[N, 0, 0]$ are physically important, because we are dealing with a system of bosons.

There are two group chains containing the angular momentum group $SO(2)$. The first one describes the vibrational motion:

$$\begin{aligned} &SU(3) \supset SU(2) \supset SO(2), \\ &|N, \qquad n_d, \qquad M\rangle, \\ &n_d = N, N-1, \ldots, 1, 0; \qquad \frac{M}{2} = n_d, n_d - 2, \ldots, 1 \text{ or } 0, \end{aligned} \tag{5}$$

and the second one characterizes the rotational limit:

$$\begin{aligned} &SU(3) \supset SO(3) \supset SO(2), \\ &|N, \qquad L, \qquad M\rangle, \\ &L = N, N-2, \ldots, 1, \text{ or } 0; \qquad \frac{M}{2} = L, L-1, \ldots, 1, 0. \end{aligned} \tag{6}$$

When the dynamical symmetry (5) holds, the energy eigenvalue is given by

$$E_v = A + Bn_d + Cn_d^2 + DM^2, \tag{7}$$

while the allowed $E2$ transitions are

$$\langle N, n_d + 1, M \pm 2 | Q_\pm | N, n_d, M\rangle = \sqrt{(N - n_d)(n_d/2 \pm M/4 + 1)}, \tag{8}$$

$$\langle N, n_d - 1, M \pm 2 | Q_\pm | N, n_d, M\rangle = \sqrt{(N - n_d + 1)(n_d/2 \mp M/4)}. \tag{9}$$

Here the transition operator is that of the two-dimensional quadrupole operator:

$$Q_\pm = a_\pm^\dagger a_0 + a_0^\dagger a_\pm. \tag{10}$$

For the rotational limit of (6) the corresponding analytical formulae are

$$E_r = a + bL(L+1) + dM^2, \tag{11}$$

$$\langle N, L, M \pm 2 | Q_\pm | N, L, M\rangle = \tfrac{1}{\sqrt{2}}\sqrt{(L \mp M/2)(L \pm M/2 + 1)}. \tag{12}$$

Typical vibrational and rotational spectra are shown on the right- and left-hand sides of figure 1, respectively. The parameters of the energy formulae are $A = 0.0$, $B = 1.0$, $C = 0.1$, $D = 0.05$, and $a = 9.6$, $b = -0.2286$, $d = 0.1167$. The parameters were chosen in such a way that the endpoints of the spectra are at the same positions. The strengths of the reduced electric transitions (absolute squares of the matrix elements) are also shown in arbitrary units. The interband transitions disappear in the rotational limit.

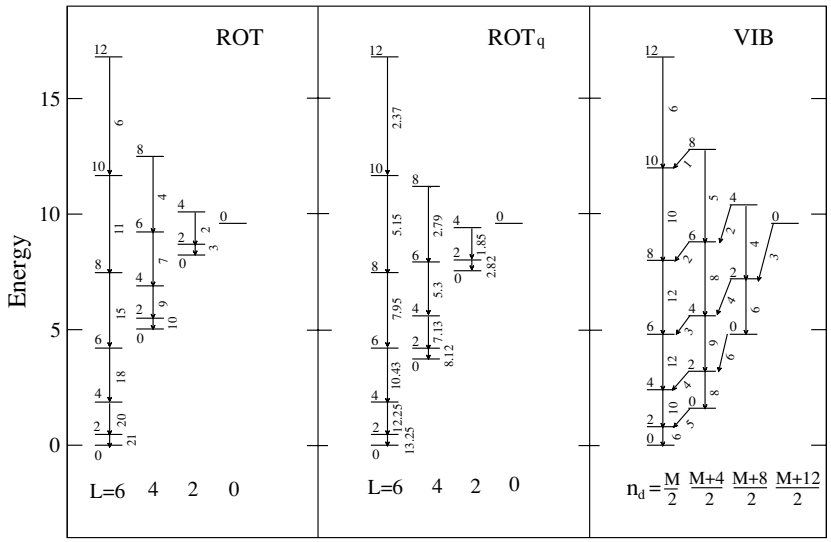

Figure 1. Classical rotational (left side) and vibrational (right side) spectrum of the two-dimensional collective model. The central part shows the best approximation (phase-deformed rotational spectrum) which can be reached by performing algebraic deformation of one classical limit towards the other one.

2.2. Quantum-deformed algebraic description

In the quantum-deformed algebraic description the

$$A_{ij} = a_i^\dagger a_j \tag{13}$$

operators close under the q-commutation relations defined as

$$[A, B]_q = AB - qBA, \tag{14}$$

see e.g. [1]. The group chain corresponding to the vibrational limit of (5) is

$$SU_q(3) \supset SU_q(2) \supset SO_q(2), \tag{15}$$

while the analogue of (6) for the completely symmetric irrep, what we need here, is

$$SU_q(3) \supset SO_q(3) \supset SO_q(2). \tag{16}$$

The quantum numbers are similar to those of the classical limits.

In the vibrational limit the matrix elements giving the energy eigenvalues are

$$E_q = A_q + B_q \left[\frac{n_d}{2}\right]\left[\frac{n_d}{2}+1\right] + D_q M^2, \tag{17}$$

while the transitions are determined by the values

$$\langle N, n_d+1, M \pm 2|Q_\pm|N, n_d, M\rangle_q = \sqrt{[N-n_d][n_d/2 \pm M/4+1]}, \tag{18}$$

$$\langle N, n_d-1, M \pm 2|Q_\pm|N, n_d, M\rangle_q = \sqrt{[N-n_d+1][n_d/2 \mp M/4]}, \tag{19}$$

and the selection rules are similar to that of the classical limit.

Here

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{20}$$

Table 1. The similarity (W) of the quantum-deformed spectrum to those of the classical dynamical symmetries ($W = 1$: no improvement at all, $W = 0$: reaching the other Lie-algebraic limit). The results for the energies and $E2$-transitions are indicated separately. The deformations should be read from left to right, e.g. rot-p(r)-vib means: starting from rotation, going towards vibration via phase (real) deformation.

Deformation	Energy	Transition
vib-p-rot	0.945	1.000
vib-r-rot	1.000	0.981
rot-p-vib	0.815	0.857
rot-r-vib	1.000	1.000

When q is real ($q = \mathrm{e}^{\tau}$, with τ real) the q-numbers can be written as

$$[x] = \frac{\sinh \tau x}{\sinh \tau}, \tag{21}$$

while when q is a phase ($q = \mathrm{e}^{\mathrm{i}\tau}$, with τ real) the q-numbers have the form

$$[x] = \frac{\sin \tau x}{\sin \tau}. \tag{22}$$

In the case of the $SO_q(3)$ dynamical symmetry, the energy eigenvalue is given by

$$E = a_q + b_q[L][L+1] + d_q M^2, \tag{23}$$

while the non-zero E2 matrix elements are

$$\langle N, L, M \pm 2|Q_{\pm}|N, L, M\rangle = \tfrac{1}{\sqrt{2}}\sqrt{[L \mp M/2][L \pm M/2 + 1]}. \tag{24}$$

3. Calculations

When approaching the rotational (vibrational) spectrum via the quantum deformation of the vibrational (rotational) one, we measure the distance from the endpoints by the quantity

$$W = \frac{\sum_i \left|E_i^B - E_i^Q\right|}{\sum_i \left|E_i^B - E_i^A\right|}, \tag{25}$$

where E_i^A are the energy values of the initial spectrum, E_i^B are the energy values of the destination spectrum and E_i^Q are the energy values of the q-deformed spectrum. Therefore, $W = 0$ indicates that the quantum deformation of the algebraic structure can completely cover the distance between the classical limits, while $W = 1$ shows a situation when the q-algebraic dynamical symmetry cannot get closer to the destination than the classical one (i.e. our starting point).

When considering the electromagnetic transitions we use a similar quantity as a measure of the distance of the quantum-deformed dynamical symmetry from the desired destination, in which case, E_i stands for the E2 transition probability, instead of the energy.

When performing the deformation of the algebraic structure the parameters of the Hamiltonian are changed in such a way that the three extremal states of the spectrum are fixed at the same positions. This requirement determines the parameters a_q, b_q, d_q or A_q, B_q, D_q in each step. (Therefore, the rotational or vibrational nature of the spectrum is reflected by its details, not by e.g. the total energy range of the ground-state band.)

Table 1 shows the results of our calculations for the four cases of the well-established q-deformed model (with real physical quantities): deformation of the vibrational and rotational

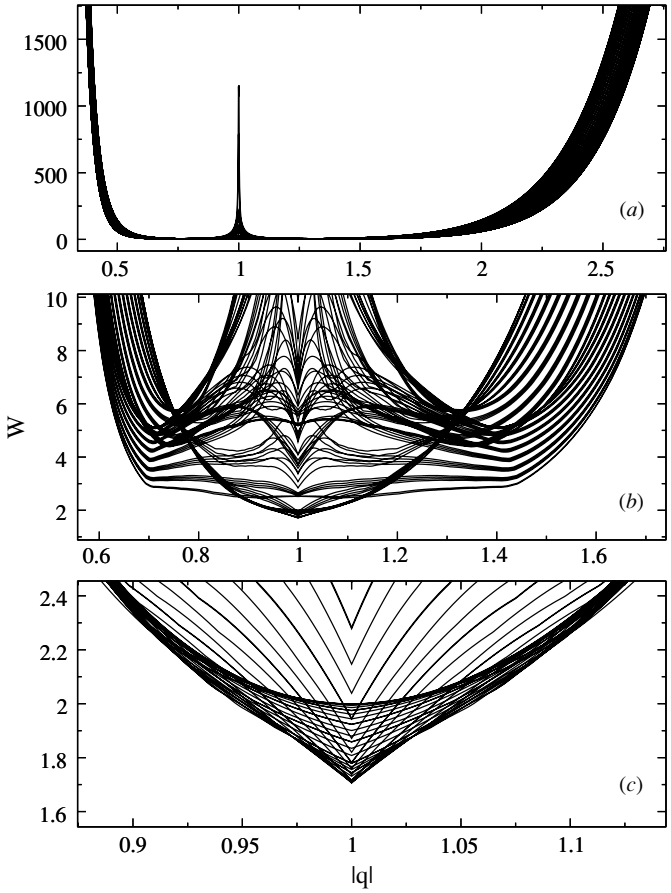

Figure 2. The similarity of the q-deformed rotational spectrum to the classical vibrational one measured by W of equation (26, 27) for arbitrary q-values. Each line corresponds to a definite phase-value of the complex number, while its absolute value is shown on the horizontal axis.

spectrum with pure phase and real parameters. In the first step we have investigated the deformation of the energy spectrum and transition ratios separately. As is seen, there is only one case in which the quantum deformation takes place in the right direction, i.e. governs the system from one of the classical dynamical symmetries towards the other, both for the energies and for the transition ratios. This is the case of the phase-deformed rotational spectrum.

In the second step of the calculation we looked for the minimum of the

$$W = W^{\text{energy}} + W^{\text{transition}} \tag{26}$$

quantity, as a function of τ. The best fit, shown in the central part of figure 1, is provided by $\tau = 0.178$, which resulted in $W = 1.705$. (In this combined fit $W = 0$ would correspond to reaching the classical vibrational limit, while $W = 2$ indicates the position of the starting point (rotational limit of the classical Lie-algebraic model).)

We have also investigated the possibility of quantum deformation with arbitrary (not real or pure phase) complex numbers. This kind of generalization involves highly non-trivial questions concerning the physical interpretation of the model, since several physical quantities may have complex (or undefined) values. Yet, based on the conjecture of [4], it seemed to be interesting to test this possibility from the purely formal viewpoint. In particular, we intended

to check what kind of improvement can be made in this manner when the search is performed systematically, and the calculation includes the whole spectrum.

The energy expression of equation (17) for the q-deformed vibrational limit is ambiguous due to the half-integer powers of the q-numbers $[\frac{n_d}{2}]$ according to equation (20). Therefore, we have looked at the q-deformation of the rotational limit, where this problem does not appear, and furthermore, this case gave the best result with the better established version of the model of real or phase deformation.

As a measure of the distance of the q-deformed spectrum from the classical limit, we have applied again a summed quantity for the energy and the transition probabilities: $W = W^{\text{energy}} + W^{\text{transition}}$. The latter part was the same as before; however, due to the complex energy-eigenvalues W^{energy} had to be modified. We calculated

$$W^{\text{energy}} = \frac{\sum_i \left|E_i^B - E_i^Q\right|}{\sum_i \left|E_i^B - E_i^A\right|} + \frac{\sum_i |\Gamma_i|}{\sum_i \left|E_i^B - E_i^A\right|}, \tag{27}$$

where Γ_i stands for the imaginary part of the energy for the ith state. This quantity has a value 1 at the starting point, i.e. at the classical rotational limit, 0 at the destination, i.e. at the classical vibrational limit, but it may have values larger than 1 in between. We have performed a systematic search for the deformation parameter, which gives the best approximation to the vibrational limit. The result is shown in figure 2, where W is shown as a function of the absolute value of the complex number, and each single line corresponds to a given phase (which was changed also systematically between 0 and 2π). The upper, middle and lower parts show the landscape, intermediate and fine-resolution pictures around the minimum. It is very interesting that allowing an arbitrary complex deformation of the algebraic structure did not give any better agreement than the pure phase-deformed model.

4. Summary and conclusions

In this paper we have addressed the question whether or not the gap between the dynamical symmetries of a Lie-algebraic model can be bridged by the analytical solutions of the quantum-deformed algebraic model. As a specific example we have investigated the two-dimensional interacting boson model, which is simple enough, having an algebraic structure of $U(3)$, but is not trivial, having two dynamical symmetries, corresponding to vibrational and rotational spectra. We have investigated the quantum deformation of both limits, and the complete spectrum, including electromagnetic transitions.

In addition to real, and pure phase values of the deformation parameter, which result in real physical quantities for the quantum-algebraic model, we have also investigated the deformation of the rotational spectrum with arbitrary complex q-numbers, which results in well-defined (complex) energies. The distance from the classical limits was measured quantitatively. It turned out that quantum-deformed dynamical symmetries cannot take the system from one classical limit to the other; in contrast, they can cover only a tiny part of the gap between them. Only the phase-deformed rotational limit could govern the system towards the other classical limit, but its similarity is only slightly better (14.7%) than that of the two classical limits. Interestingly enough it turned out that the arbitrary complex q-deformation did not improve the agreement that can be obtained with pure phase deformation.

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